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A NOTE ON THE J_1-CONVERGENCE OF A FIRST-PASSAGE PROCESS

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Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables with mean zero such that the common distribution function belongs to the domain of attraction of a stable law $G_{\alpha,\beta}$ with $1 < \alpha < 2$ and $\beta = 1$ or $\alpha = 2$. If $S_n = X_1 + \ldots + X_n$ and $N(\xi) = \min\{k : S_k > \xi\}$, $\xi \geqslant 0$, then it is shown that $N(nt)/B^*(n)$, $0 \leqslant t \leqslant 1$, converges weakly under the Skorohod J_1-topology to a stable subordinator of index $1/\alpha$, where $B^*(n)$ depends on the norming constant for S_n.

Skorohod J_1- and M_1-topologies	stable law
first-passage process	weak convergence
domain of attraction	ladder indices
stable subordinator	ladder heights
regularly varying functions	random time change

1. Introduction

Let $\{X_n, n = 1, 2, 3, \ldots\}$ be a sequence of independent and identically distributed random variables, with a common distribution function F. Let the mean of F be zero and $S_n = X_1 + X_2 + \ldots + X_n$. Suppose that there exists a sequence of positive constants $B(n)$, $n = 1, 2, \ldots,$ such that

$$\lim_{n\to\infty} \mathrm{P}[S_n/B(n) \leqslant x] = G_{\alpha,\beta}(x)\,, \tag{1}$$

where

$$\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}tx}\, \mathrm{d}G_{\alpha,\beta}(x) = \exp\{-c\,|t|^{\alpha}(1 + \mathrm{i}\beta\, t\,|t|^{-1} \tan(\tfrac{1}{2}\pi\alpha))\}\,,$$

$$c > 0,\ 1 < \alpha \leqslant 2,\ -1 \leqslant \beta \leqslant 1\,.$$

Define

$$N(\xi) = \min\{k: S_k > \xi\}, \quad \xi \geqslant 0,$$

$$Z_n(t) = N(nt)/B^*(n), \quad 0 \leqslant t \leqslant 1,$$

where

$$\lim_{n\to\infty} \{B([B^*(n)])/n\} = \lim_{n\to\infty} \{B^*([B(n)])/n\} = 1.$$

Here $[x]$ denotes the greatest integer less than or equal to x.

In this paper the processes under consideration take values in the function space $D = D[0, 1]$ of real-valued functions on [0,1] which are right continuous with left limits, and D is endowed with either the Skorohod J_1-topology or the weaker Skorohod M_1-topology.

Bingham [2] has shown that the process $Z_n(t)$ converges weakly in the Skorohod M_1-topology to the process

$$Z(t) = \inf\{u: X(u) > t\}, \quad 0 \leqslant t \leqslant 1,$$

where

$$\mathrm{E}\,\mathrm{e}^{\mathrm{i}uX(t)} = \exp\{-ct|u|^\alpha(1 + \mathrm{i}\beta u|u|^{-1}\tan(\tfrac{1}{2}\pi\alpha))\},$$
$$c > 0,\ 1 < \alpha \leqslant 2,\ -1 \leqslant \beta \leqslant 1. \tag{2}$$

In this note we prove that when $1 < \alpha < 2$ and $\beta = 1$ or $\alpha = 2$, $Z_n(t)$ converges weakly to $Z(t)$ under the Skorohod J_1-topology, where by [2, Proposition 1], $Z(t)$ now satisfies

$$\mathrm{E}\{\mathrm{e}^{-sZ(t)}\} = \exp\{-t(s/c_1)^{1/\alpha}\}, \quad s \geqslant 0, \tag{3}$$

where $c_1 = c(1 + \tan^2(\frac{1}{2}\pi\alpha))^{1/2}$, that is, $Z(t)$ is a stable subordinator of index $1/\alpha$.

It is interesting to note that when $1 < \alpha < 2$ and $\beta = 1$ or $\alpha = 2$ and F has positive mean μ, Bingham [2] has shown that the process $((N(nt) - nt\mu^{-1})/B(n))\,\mu^{(1+\alpha)/\alpha}$ converges to $-X(t)$ weakly under the M_1-topology, where $X(t)$ is given by (2), while Gut [4] established the J_1-convergence. It may be remarked here that the random time change is possible in the cases under consideration mainly because of Lemma 2.1. In other cases ($0 < \alpha < 2$ and $\beta < 1$), Y_1 (see Section 2) belongs to the domain of attraction of a spectrally positive stable law (see [6]) and hence (5) is no longer true for any sequence $A_1(n)$. This in turn implies that in Lemma 2.1 convergence in probability is impossible.

2. Some lemmas

In what follows, unless stated otherwise, let S_n satisfy (1) with $1 < \alpha < 2$ and $\beta = 1$ or $\alpha = 2$. Let

$$N_1, \quad N_1 + N_2, \quad N_1 + N_2 + N_3, \ldots$$

be the strong ascending ladder indices, that is,

$$N_1 = \min\{k: S_k > 0\}, \quad N_1 + N_2 = \min\{k > N_1: S_k > S_{N_1}\}, \ldots,$$

and let

$$Y_1 = S_{N_1}, \quad Y_1 + Y_2 = S_{N_1+N_2}, \ldots$$

be the corresponding ladder heights. Then $N_1, N_2, N_3, \ldots$ and $Y_1, Y_2, Y_3 \ldots$ are two sequences of independent identically distributed positive random variables (see [3]).

Define the first-passage time

$$Q(\xi) = \min\{k: \Sigma_{j=1}^{k} Y_j > \xi\}, \quad \xi \geqslant 0.$$

Since the first passage must occur at a ladder point, we have

$$Z_n(t) = (N_1 + N_2 + \ldots + N_{Q(nt)})/B^*(n), \quad 0 \leqslant t \leqslant 1. \tag{4}$$

From [6, Theorem 9] it follows that there exists a sequence of numbers $A_1(n) > 0$, $n = 1, 2, \ldots$, such that

$$(Y_1 + Y_2 + \ldots + Y_n)/A_1(n) \xrightarrow{p} 1 \tag{5}$$

($\xrightarrow{p}$ denotes convergence in probability); that is, Y_1 is relatively stable. Further, from [6, Theorem 2] it follows that (5) is equivalent to

$$\int_0^x \mathbf{P}[Y_1 > y]\,\mathrm{d}y \sim L(x) \quad \text{as } x \to \infty, \tag{6}$$

where L is some slowly varying function and the sign $\sim$ links functions whose ratio when passing to the limit as stated tends to 1. Also, the constants $A_1(n)$ satisfy

$$L(A_1(n))/A_1(n) \sim 1/n \quad \text{as } n \to \infty. \tag{7}$$

Let $0 < \theta < 1$ and $A(n) = \theta A_1(n)$. (*Note*: Our method of proof involves a random time change defined in terms of θ. However, θ is held constant throughout, and we use it only at the end of the proof.) Then from (5) and (7) we have, as $n \to \infty$,

$$(Y_1 + Y_2 + \dots + Y_n)/A(n) \xrightarrow{p} 1/\theta, \qquad L(A(n))/A(n) \sim 1/\theta n. \quad (8)$$

Lemma 2.1. *As $n \to \infty$, $Q(n)/A^*(n) \xrightarrow{p} \theta$, where*

$$A([A^*(n)]) \sim A^*([A(n)]) \sim n. \quad (9)$$

Proof. By the definition of Q we have for $\epsilon > 0$,

$$\mathbf{P}[Q(n) > (\theta+\epsilon)A^*(n)] = \mathbf{P}\left[\frac{\sum_{k=1}^{[A^*(n)(\theta+\epsilon)]} Y_k}{A([A^*(n)(\theta+\epsilon)])} \leqslant \frac{n}{A([A^*(n)(\theta+\epsilon)])}\right].$$

From (8) it is easy to see that $A(n)$ is regularly varying at infinity with exponent 1, and hence, as $n \to \infty$,

$$\frac{n}{A([A^*(n)(\theta+\epsilon)])} \sim \frac{n}{(\theta+\epsilon)A([A^*(n)])} \sim \frac{1}{\theta+\epsilon},$$

and therefore

$$\lim_{n\to\infty} \mathbf{P}\left[\frac{Q(n)}{A^*(n)} > \theta+\epsilon\right] = \lim_{n\to\infty} \mathbf{P}\left[\frac{\sum_{k=1}^{[A^*(n)(\theta+\epsilon)]} Y_k}{A([A^*(n)(\theta+\epsilon)])} \leqslant \frac{1}{\theta+\epsilon}\right] = 0. \quad (10)$$

Similarly it can be shown that

$$\lim_{n\to\infty} \mathbf{P}\left[\frac{Q(n)}{A^*(n)} < \theta - \epsilon\right] = 0. \quad (11)$$

The lemma follows from (10) and (11). □

Since $A^*(n)$ is also regularly varying with exponent 1, we conclude from Lemma 2.1 that for each t, $0 \leqslant t \leqslant 1$,

$$Q(nt)/A^*(n) \xrightarrow{p} t\theta. \quad (12)$$

From (4) and [2, Theorem 2] we have

$$\lim_{n\to\infty} \mathbf{P}[(N_1 + \ldots + N_{Q(n)})/B^*(n) \leqslant x] = \lim_{n\to\infty} \mathbf{P}[Z_n(1) \leqslant x]$$

$$= H_1(x), \quad x > 0, \tag{13}$$

where from (3), $H_1(x)$ has the Laplace–Stieltjes transform

$$\exp\{-(s/c_1)^{1/\alpha}\}, \quad s \geqslant 0.$$

Lemma 2.2. $\lim_{n\to\infty} \mathbf{P}[(N_1 + \ldots + N_{[\theta A^*(n)]})/B^*(n) \leqslant x] = H_1(x), x \geqslant 0.$

Proof. From Lemma 2.1 and (13) wehave for $\epsilon > 0$,

$$\lim_{n\to\infty} \mathbf{P}\left[\frac{N_1 + \ldots + N_{Q(n)}}{B^*(n)} \leqslant x \,,\, \left|\frac{Q(n)}{\theta A^*(n)} - 1\right| \leqslant \epsilon\right] = H_1(x).$$

Since $N_1, N_2, \ldots$ are postive-valued random variables, we get

$$\mathbf{P}\left[\frac{N_1 + \ldots + N_{[\theta A^*(n)(1+\epsilon)]}}{B^*(n)} \leqslant x, \left|\frac{Q(n)}{\theta A^*(n)} - 1\right| \leqslant \epsilon\right] \leqslant$$

$$\leqslant \mathbf{P}\left[\frac{N_1 + \ldots + N_{Q(n)}}{B^*(n)} \leqslant x, \left|\frac{Q(n)}{\theta A^*(n)} - 1\right| \leqslant \epsilon\right]$$

$$\leqslant \mathbf{P}\left[\frac{N_1 + \ldots + N_{[\theta A^*(n)(1-\epsilon)]}}{B^*(n)} \leqslant x, \left|\frac{Q(n)}{\theta A^*(n)} - 1\right| \leqslant \epsilon\right].$$

From this it follows that

$$\mathbf{P}\left[\frac{N_1 + \ldots + N_{[\theta A^*(n)(1+\epsilon)]}}{B^*(n)} \leqslant x\right] - \mathbf{P}\left[\left|\frac{Q(n)}{\theta A^*(n)} - 1\right| > \epsilon\right] \leqslant$$

$$\leqslant \mathbf{P}\left[\frac{N_1 + \ldots + N_{Q(n)}}{B^*(n)} \leqslant x, \left|\frac{Q(n)}{\theta A^*(n)} - 1\right| \leqslant \epsilon\right]$$

$$\leqslant \mathbf{P}\left[\frac{N_1 + \ldots + N_{[\theta A^*(n)(1-\epsilon)]}}{B^*(n)} \leqslant x\right].$$

Taking the limit as $n \to \infty$, we obtain

$$\limsup_{n\to\infty} \mathbf{P}\left[\frac{N_1 + \ldots + N_{[\theta A^*(n)(1+\epsilon)]}}{B^*(n)} \leqslant x\right] \leqslant$$

$$\leqslant H_1(x) \leqslant \liminf_{n\to\infty} \mathbf{P}\left[\frac{N_1 + \ldots + N_{[\theta A^*(n)(1-\epsilon)]}}{B^*(n)} \leqslant x\right].$$

Using the fact that $B^*(n)$ is regularly varying with exponent α (since $B(n)$ is so with exponent $1/\alpha$) and $A^*(n)$ is regularly varying with exponent 1, we obtain

$$\limsup_{n\to\infty} \mathbf{P}\left[\frac{N_1 + \ldots + N_{[\theta A^*(n)(1+\epsilon)]}}{B^*(n)} \leqslant x\right] =$$

$$= \limsup_{n\to\infty} \mathbf{P}\left[\frac{N_1 + \ldots + N_{[\theta A^*(n)]}}{B^*(n)} < \frac{x}{(1+\epsilon)^\alpha}\right] \leqslant H_1(x).$$

This can be written as

$$\limsup_{n\to\infty} \mathbf{P}\left[\frac{N_1 + \ldots + N_{[\theta A^*(n)]}}{B^*(n)} \leqslant y\right] \leqslant H_1(y(1+\epsilon)^\alpha).$$

Letting $\epsilon \to 0$, we have finally

$$\limsup_{n\to\infty} \mathbf{P}\left[\frac{N_1 + \ldots + N_{[\theta A^*(n)]}}{B^*(n)} \leqslant y\right] \leqslant H_1(y)$$

since H_1 is continuous. Similarly,

$$\liminf_{n\to\infty} \mathbf{P}\left[\frac{N_1 + \ldots + N_{[\theta A^*(n)]}}{B^*(n)} \leqslant y\right] \geqslant H_1(y),$$

and the proof of the lemma is complete. □

Since the mean of X_1 is zero, we have from [5, ch. 6, Theorem 1.6] that

$$\sum_{n=1}^{\infty} (1/n)\mathbf{P}[S_n > 0] = \infty.$$

Further

$$\lim_{n\to\infty} (1/n) \sum_{k=1}^{n} \mathbf{P}[S_k > 0] = \lim_{n\to\infty} \mathbf{P}[S_n > 0] = 1 - G_{\alpha\beta}(0) = \rho \text{ (say)}.$$

Hence from [6, Theorem 4] N_1 belongs to the domain of attraction of a positive stable law $H_2(x)$ with exponent ρ, that is, there exists a sequence of positive constants $C_1(n)$, $n = 1, 2, \dots$, such that

$$\lim_{n\to\infty} \mathbf{P}[(N_1 + \dots + N_n)/C_1(n) \leqslant x] = H_2(x), \quad x \geqslant 0. \tag{14}$$

It may be noted that in our case $\rho = 1/\alpha$ since in general $\rho = 1 - G_{\alpha,\beta}(0) = \frac{1}{2} + (1/\pi\alpha) \arctan(-\beta \tan(\frac{1}{2}\pi\alpha))$, which is equal to $1/\alpha$ if and only if $1 < \alpha < 2$ and $\beta = 1$ or $\alpha = 2$ (see for example [2]). Further, we notice that the two stable random variables with distribution functions $H_1(x)$ and $H_2(x)$ above differ only by a scalar multiplier. Hence in (14) it is possible to choose a new sequence of constants $C(n)$, $n = 1, 2, \dots$, which differ from $C_1(n)$, $n = 1, 2, \dots$, only by a scalar multiplier independent of n, to obtain

$$\lim_{n\to\infty} \mathbf{P}[(N_1 + \dots + N_n)/C(n) \leqslant x] = H_1(x), \quad x \geqslant 0. \tag{15}$$

An immediate consequence of Lemma 2.2 and (15) is that, as $n \to \infty$,

$$B^*([A(n)/\theta]) \sim C(n). \tag{16}$$

Lemma 2.3. *If $U_n(t) = (Y_1 + Y_2 + \dots + Y_{[nt]})/A(n)$, $0 \leqslant t \leqslant 1$, then $U_n(t)$ converges weakly to $U(t) = t/\theta$ under the J_1-topology.*

Proof. From (8) and the fact that $A(n)$ is regularly varying with exponent 1 it follows that $U_n(t)$ converges in distribution to $U(t)$ for each t. The lemma follows now by a straightforward application of [7, Theorem 2.7].

3. The main theorem

Theorem 3.1. *Let S_n satisfy* (1) *with $1 < \alpha < 2$ and $\beta = 1$ or $\alpha = 2$. Then $Z_n(t)$ converges to the process $Z(t)$ given by* (3) *weakly under the Skorohod J_1-topology.*

Proof. From [7, Theorem 2.7] it follows that the process

$$Z_n^*(t) = (N_1 + N_2 + \dots + N_{[nt]})/C(n)$$

converges weakly in the J_1-topology to the process $Z(t)$ given by (3). In view of (16), this implies that the process

$$Z_n^{**}(t) = \frac{N_1 + N_2 + \ldots + N_{[t\theta A^*(n)]}}{B^*(n)}$$

converges to the same limit process $Z(t)$ weakly under the J_1-topology. Define the random change of time

$$\phi_n(t) = \begin{cases} \dfrac{Q(nt)}{\theta A^*(n)} & \text{if } \dfrac{Q(n)}{A^*(n)} \leqslant 1, \\ t & \text{otherwise.} \end{cases}$$

In what follows, convergence in probability will be in the sense of random elements, as in [1, p. 24]; here the metric ρ is a metric generating the J_1-topology, as in [1, ch. 3]. The first step is to prove that $\phi_n(t)$ converges in probability to $\phi(t) = t$. To accomplish this, note that

$$\frac{Q(tA(n))}{n} = \inf\{u \colon U_n(u) > t\}.$$

From Lemma 2.3, since the first-passage time function is M_1-continuous (see [8]), we conclude that $Q(tA(n))/n$ converges weakly to $\phi^*(t) = t\theta$ under the Skorohod M_1-topology. Since the limit process (degenerate) is continuous, $Q(tA(n))/n$ converges weakly under the J_1-topology (see [1, p. 151]). Further, since $\phi^*(t)$ is a constant-valued radnom element, weak convergence is equivalent to convergence in probability (see [1, p. 25]). Thus we have shown that $Q(tA(n))/n$ converges in probability to $\phi^*(t)$. Finally, $Q(tA(n))/n$ and $Q(tn)/A^*(n)$ converge to the same limit process $\phi^*(t)$, and this proves the assertion that $\phi_n(t)$ converges in probability to $\phi(t) = t$.

From [1, Theorem 4.4], (Z_n^{**}, ϕ_n) converges weakly to (Z, ϕ) relative to the product topology. Since the composition $x \circ y = x(y(t))$ is continuous at (x, y) if $x \in D$ and y is a strictly increasing continuous function (see [8]), $(Z_n^{**} \circ \phi_n)(t)$ converges to $(Z \circ \phi)(t) = Z(t)$ weakly under the J_1-topology. But $(Z_n^{**} \circ \phi_n)(t) = Z_n(t)$ if $Q(n)/A^*(n) \leqslant 1$, the probability of which goes to 1 since $\theta < 1$. This establishes the theorem. □

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